On the Adjacency Matrix and Neighborhood Associated with Zero-divisor Graph for Direct Product of Finite Commutative Rings

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Abstract: The main purpose of this paper is to study the zero-divisor graph for direct product of finite commutative rings. In our present investigation we discuss the zero-divisor graphs for the following direct products: direct product of the ring of integers under addition and multiplication modulo \( p \) and the ring of integers under addition and multiplication modulo \( p^2 \) for a prime number \( p \), direct product of the ring of integers under addition and multiplication modulo \( p \) and the ring of integers under addition and multiplication modulo \( 2p \) for an odd prime number \( p \) and direct product of the ring of integers under addition and multiplication modulo \( p^2 \) for that odd prime \( p \) for which \( p^2 - 2 \) is a prime number. The aim of this paper is to give some new ideas about the neighborhood, the neighborhood number and the adjacency matrix corresponding to zero-divisor graphs for the above mentioned direct products. Finally, we prove some results of annihilators on zero-divisor graph for direct product of \( A \) and \( B \) for any two commutative rings \( A \) and \( B \) with unity.

Keywords: Zero-divisor, Commutative ring, Adjacency matrix, Neighborhood, Zero-divisor graph, Annihilator.

AMS Classification (2010): 05Cxx; 05C25; 05C50.

1. INTRODUCTION


In this paper \( R_1 \) denotes the finite commutative ring such that \( R_1 = \mathbb{Z}_p \times \mathbb{Z}_p \) (\( p \) is a prime number), \( R_2 \) denotes the finite commutative ring such that \( R_2 = \mathbb{Z}_p \times \mathbb{Z}_{p^2} \) (\( p \) is an odd prime number) and \( R_3 \) denotes the finite commutative ring such that \( R_3 = \mathbb{Z}_p \times \mathbb{Z}_{p^2 - 2} \) (for that odd prime \( p \) for which \( p^2 - 2 \) is a prime number). Let \( R \) be a commutative ring with unity and \( \mathbb{Z}(R) \) be the set of zero-divisors of \( R \); that is \( \mathbb{Z}(R) = \{ x \in R : xy = 0 \ or \ yx = 0 \ for \ some \ y \in R^* = R - \{0\} \} \). Then zero-divisor graph of \( R \) is an undirected graph \( \Gamma(R) \) with vertex set \( \mathbb{Z}(R)^* = \mathbb{Z}(R) - \{0\} \) such that distinct vertices \( x \) and \( y \) of \( \mathbb{Z}(R)^* \) are adjacent if and only if \( xy = 0 \). The neighborhood (or open neighborhood) \( N_G(v) \) of a vertex \( v \) of a graph \( G \) is the set of vertices adjacent to \( v \). The closed neighborhood \( N_G[v] \) of a vertex \( v \) is the set \( N_G(v) \cup \{v\} \). For a set \( S \) of vertices, the neighborhood of \( S \) is the union of the neighborhoods of the vertices and so it is the set of all vertices adjacent to at least one member of \( S \). For a graph \( G \) with vertex set \( V \), the union of the neighborhoods of all the vertices is neighborhood of \( V \) and it is denoted by \( N_G(V) \). The neighborhood number \( n_G(V) \) is the cardinality of \( N_G(V) \). If the graph \( G \) with vertex set \( V \) is connected, then \( N_G(V) \) is the vertex set \( V \) and the cardinality of \( N_G(V) \) is equal to the cardinality of \( V \). If \( \Gamma(R) \) is the zero-divisor graph of a commutative ring \( R \) with vertex set \( \mathbb{Z}(R)^* \) and since zero-divisor graph is always connected \([1]\), we have \( N_{\Gamma(R)}(\mathbb{Z}(R)^*) = \mathbb{Z}(R)^* \) and \( |N_{\Gamma(R)}(\mathbb{Z}(R)^*)| = |\mathbb{Z}(R)^*| \). Throughout this paper \( \Delta(G) \) denotes the maximum degree of a graph \( G \) and \( \delta(G) \) denotes the minimum degree of a graph \( G \). The adjacency matrix corresponding to zero-divisor graph of \( G \) is defined as \( A = [a_{ij}] \), where \( a_{ij} = 1 \), if \( v_i, v_j \in 0 \) for any vertex \( v_i \) and \( v_j \) of \( G \) and \( a_{ij} = 0 \), otherwise.

In this paper, we construct zero-divisor graphs for the rings \( R_1, R_2 \) and \( R_3 \). We obtain the neighborhood and the adjacency matrices corresponding to zero-divisor graphs of \( R_1, R_2 \) and \( R_3 \). Some properties of adjacency matrices are also obtained. We prove some theorems related to neighborhood and adjacency matrices corresponding to zero-divisor graphs of \( R_1, R_2 \) and \( R_3 \). Finally, we prove some results of annihilators on zero-divisor graph of \( A \times B \), for any two commutative rings \( A \) and \( B \) with unity.
2. CONSTRUCTION OF ZERO-DIVISOR GRAPH FOR \( R_I = Z_p \times Z_p \): (p is a prime number):

First, we construct the zero-divisor graph for the ring \( R_I = Z_p \times Z_p \). We start with the cases \( p = 2 \) and \( p = 3 \) and then generalize the cases.

**Case 1**: When \( p = 2 \) we have \( R_I = Z_2 \times Z_2 \).
The ring \( R_I \) has 5 non-zero zero-divisors. In this case \( V = Z(R_I)^* = \{(1,0), (0,1), (0,2), (0,3), (1,2)\} \) and the zero-divisor graph \( G = \Gamma(R_I) \) is given by:

![Zero-Divisor Graph](image)

The closed neighborhoods of the vertices are \( N_G[(1,0)] = \{(1,0), (0,1), (0,2), (0,3), (1,2)\} = \{(1,0), (0,1)\} \), \( N_G[(0,2)] = \{(1,0), (1,2), (0,2)\} \), \( N_G[(0,3)] = \{(1,0), (0,3)\} \), and \( N_G[(1,2)] = \{(0,2), (1,2)\} \). The neighborhood of \( V \) is given by \( N_G(V) = \{(1,0), (0,1), (0,2), (0,3), (1,2)\} \). The maximum degree is \( \Delta(G) = 3 \) and minimum degree is \( \delta(G) = 1 \). The adjacency matrix for the zero-divisor graph of \( R_I = Z_2 \times Z_2 \) is

\[
M_I = \begin{bmatrix}
0 & A_{1 \times 3} & 0 \\
0 & B_{1 \times 3} & 0 \\
\end{bmatrix}
\]

where, \( A_{1 \times 3} = [1 \ 1 \ 1] \), \( B_{1 \times 3} = [1 \ 1 \ 0] \), \( A_{1 \times 3}^T \) is the transpose of \( A_{1 \times 3} \), \( B_{1 \times 3}^T \) is the transpose of \( B_{1 \times 3} \) and \( O_{3 \times 3} \) is the zero matrix.

**Properties of adjacency matrix \( M_I \):**

(i) The determinant of the adjacency matrix \( M_I \) corresponding to \( G = \Gamma(R_I) \) is 0.
(ii) The rank of the adjacency matrix \( M_I \) corresponding to \( G = \Gamma(R_I) \) is 2.

**Case 2**: When \( p = 3 \) we have \( R_I = Z_3 \times Z_3 \).
The ring \( R_I \) has 14 non-zero zero-divisors. In this case \( V = Z(R_I)^* = \{(1,0), (2,0), (1,3), (1,6), (2,3), (2,6), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8)\} \) and the zero-divisor graph \( G = \Gamma(R_I) \) is given by:

![Zero-Divisor Graph](image)

The closed neighborhoods of the vertices are \( N_G[(1,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8)\} \), \( N_G[(2,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8)\} \), \( N_G[(1,3)] = \{(0,3), (0,6), (1,3)\} \), \( N_G[(1,6)] = \{(0,3), (0,6), (1,6)\} \), \( N_G[(2,3)] = \{(0,3), (0,6), (2,3)\} \), \( N_G[(2,6)] = \{(0,3), (0,6), (2,6)\} \), \( N_G[(1,0)] = \{(1,0), (2,0), (0,1)\} \), \( N_G[(0,2)] = \{(1,0), (2,0), (2,2)\} \), \( N_G[(0,3)] = \{(1,0), (2,0), (2,3)\} \), \( N_G[(0,4)] = \{(1,0), (2,0), (2,4)\} \), \( N_G[(0,5)] = \{(1,0), (2,0), (2,5)\} \), \( N_G[(0,6)] = \{(1,0), (2,0), (2,6)\} \), \( N_G[(0,7)] = \{(1,0), (2,0), (2,7)\} \), \( N_G[(0,8)] = \{(1,0), (2,0), (2,8)\} \). The neighborhood of \( V \) is given by \( N_G(V) = \{(1,0), (0,2), (1,3), (1,6), (2,3), (2,6), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8)\} \). The maximum degree is \( \Delta(G) = 8 \) and minimum degree is \( \delta(G) = 2 \). The adjacency matrix for the zero-divisor graph of \( R_I = Z_3 \times Z_3 \) is

\[
M_I = \begin{bmatrix}
O_{6 \times 6} & A_{6 \times 5} & B_{6 \times 3} \\
A_{5 \times 6}^T & O_{5 \times 5} & C_{5 \times 3} \\
B_{3 \times 6}^T & C_{3 \times 5}^T & O_{3 \times 3}
\end{bmatrix}
\]

where

\[
A_{6 \times 5} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B_{6 \times 3} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

the zero matrices and \( A_{5 \times 6}^T, B_{3 \times 6}^T, C_{3 \times 5}^T \) are the transposes of \( A_{6 \times 5}, B_{6 \times 3}, C_{5 \times 3} \) respectively.

**Properties of adjacency matrix \( M_I \):**
The determinant of the adjacency matrix of $R_1$ corresponding to $G = \Gamma(R_1)$ is zero.

(ii) The rank of the adjacency matrix of $R_1$ corresponding to $G = \Gamma(R_1)$ is 2.

(iii) The adjacency matrix of $R_1$ corresponding to $G = \Gamma(R_1)$ is symmetric and singular.

**Generalization for $R_1 = Z_p \times Z_{p^2}$ ($p$ is a prime number):**

**Lemma 2.1:** The number of vertices of $G = \Gamma(Z_{p^2})$ is $p - 1$

and $G = \Gamma(Z_{p^2})$ is $K_{p^2}$, where $p$ is a prime number [4]

**Proof:** The multiples of $p$ less than $p^2$ are $p, 2p, 3p, \ldots$, $(p - 1)p$. These multiples of $p$ are the only non-zero zero-divisors of $Z_{p^2}$. If $G = \Gamma(Z_{p^2})$ is the zero-divisor graph of $Z_{p^2}$, then the vertices of $G = \Gamma(Z_{p^2})$ are the non-zero zero-divisors of $Z_{p^2}$. So, the vertex set of $G = \Gamma(Z_{p^2})$ is $\{Z \in Z_{p^2}^*, \, Z \neq 0 \}$.

Hence, the number of vertices of $G = \Gamma(Z_{p^2})$ is $p - 1$. Also, in $G = \Gamma(Z_{p^2})$, every vertex is adjacent to every other vertex. This gives $G = \Gamma(Z_{p^2})$ is $K_{p^2}$.

**Theorem 2.2:** Let $R_1$ be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ ($p$ is a prime number). Let $G = \Gamma(R_1)$ be the zero-divisor graph with vertex set $\{Z \in Z_{p^2}^*, \, Z \neq 0 \}$. Then number of vertices of $G = \Gamma(R_1)$ is $2p^2 - p - 1$, $\Delta(G) = p^2 - 1$ and $\delta(G) = p - 1$.

**Proof:** Let $R_1$ be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ ($p$ is a prime number). Let $R_1 = \Gamma(R_1) - \{0\}$. Then $R_1^*$ can be partitioned into disjoint sets $A, B, C, D$ and $E$ such that $A = \{(a, 0) : a \in Z_p^* \}$, $B = \{(0, v) : v \in Z_{p^2}^* \}$ and $v \neq Z \in Z_{p^2}^* \}$.

$D = \{(a, b) : a \in Z_p^* \, \text{and} \, b \in Z_{p^2}^* \}$ and $E = \{(c, d) : c \in Z_p^* \, \text{and} \, d \in Z_{p^2}^* \}$. Clearly, all the elements in $A, B, C$ are non-zero zero-divisors. Let $(a, b) \in D$ and $(0, v) \in E$. Here, $b \in Z_{p^2}^*$, $c \in Z_p^*$, and $d \in Z_{p^2}^*$. So, $p/b$ and $p/d$ are $p^2/3b$, $p^2/3d$. Therefore, $(a, b) (0, v) = (0, 0)$. Hence, every element of $D$ is a non-zero zero-divisor. Thus product of any two elements of $E$ is not equal to zero. Also, product of any element of $E$ with any element of $A, B, C$ and $D$ is not equal to zero because, $c \neq 0$ for $c, u \in Z_p^*$, $dv \neq 0$ for $d, v \in Z_{p^2}^* \text{ and } d, v \notin Z(Z_{p^2})^*$, $dw \neq 0$ for $d, w \in Z_{p^2}^*$ and $d \notin Z(Z_{p^2})^*$, $w \in Z(Z_{p^2})^*$.

Therefore, deg$_G(y) = p - 1$.

Hence, we have $\Delta(G) = p^2 - 1$ and $\delta(G) = p - 1$.

**Theorem 2.3:** Let $M_1$ be the adjacency matrix for the zero-divisor graph $G = \Gamma(R_1)$, then $M_1$ is symmetric and singular.
Proof: Let \( R \) be a finite commutative ring such that \( R = Z_p \times Z_{p^2} \) (\( p \) is a prime number). Let \( G = \Gamma(R) \) be the zero-divisor graph with vertex set \( V = Z(R) \) and \( M \) be the adjacency matrix for the zero-divisor graph of \( R = Z_p \times Z_{p^2} \).

(i) Since, at least two vertices of \( G = \Gamma(R) \) are adjacent to same vertex of \( G \), so \( M \) contains at least two identical rows (eg. for \( Z_2 \times Z_2 \)). Therefore, the determinant of the adjacency matrix \( M \) is zero.

(ii) Clearly \( M \) is symmetric. Since, the determinant of the adjacency matrix \( M \) is zero, \( M \) is singular.

Theorem 2.4: Let \( R \) be a finite commutative ring such that \( R = Z_p \times Z_{p^2} \) (\( p \) is a prime number). Let \( G = \Gamma(R) \) be the zero-divisor graph with vertex set \( V = Z(R) \). Then \( n_d(V) = 2 \Delta(G) - \delta(G) \), where \( n_d(V) \) is the neighborhood number, \( \Delta(G) \) and \( \delta(G) \) denote the maximum and minimum degree of \( G \) respectively.

Proof: Let \( R \) be a finite commutative ring such that \( R = Z_p \times Z_{p^2} \) (\( p \) is a prime number). Let \( G = \Gamma(R) \) be the zero-divisor graph with vertex set \( V = Z(R) \). Since, \( G = \Gamma(R) \) is connected [1], we have \( n_d(V) = |N_G(V)| = |V| - |Z(R)^{-1}| \).

But from Theorem 2.2, we have \( |Z(R)^{-1}| = 2p^2 - p - 1 \). Therefore, \( n_d(V) = 2p^2 - p - 1 \). This implies \( n_d(V) = 2(p^2 - 1) - (p - 1) \). Also, \( \Delta(G) = p^2 - 1 \) and \( \delta(G) = p - 1 \) [from Theorem 2.2]. This gives \( n_d(V) = 2 \Delta(G) - \delta(G) \).

3. CONSTRUCTION OF ZERO-DIVISOR GRAPH FOR \( R_2 = Z_p \times Z_{2p} \): (\( p \) IS AN ODD PRIME NUMBER):

Secondly, we construct the zero-divisor graph for the ring \( R_2 = Z_p \times Z_{2p} \) (\( p \) is an odd prime number) and analyze the graph. We start with the cases \( p = 3 \) and \( p = 5 \) and then generalize the cases.

Case 1: When \( p = 3 \) we have \( R_2 = Z_3 \times Z_6 \). The ring \( R_2 \) has 13 non-zero zero-divisors. In this case \( V = Z(R_2)^* = \{ (1,0),(2,0),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(0,1),(0,2),(0,3),(0,4),(0,5) \} \) and the zero-divisor graph \( G = \Gamma(R_2) \) is given by:

![Graph](image)

Fig: 3

The closed neighborhoods of the vertices are \( N_{\{1,0\}} = \{(1,0),(0,2),(0,3),(0,4),(0,5),(1,0)\}, N_{\{2,0\}} = \{(0,2),(0,3),(0,4),(0,5),(2,0)\}, N_{\{1,3\}} = \{(0,2),(0,4),(1,3)\}, N_{\{1,4\}} = \{(0,3),(1,4)\}, N_{\{2,2\}} = \{(0,3),(2,2)\}, N_{\{2,3\}} = \{(0,4),(2,3)\}, N_{\{2,4\}} = \{(0,5)\} \). The neighborhood of \( V \) is given by \( N_{\{1,0\}} = \{(1,0),(2,0),(0,1)\}, N_{\{0,2\}} = \{(1,0),(2,0),(1,3),(2,3),(3,0),(0,2)\}, N_{\{0,3\}} = \{(1,0),(2,0),(1,4),(2,2),(2,4),(0,2),(0,4),(0,3)\}, N_{\{0,4\}} = \{(1,0),(2,0),(1,3),(2,3),(0,3),(0,4)\}, N_{\{0,5\}} = \{(1,0),(2,0),(0,5)\}\). The neighborhood of the vertices are \( N_{\{1,0\}} = \{(1,0),(2,0),(1,3),(1,4),(2,2),(2,3),(2,4),(0,1),(0,2),(0,3),(0,4),(0,5)\}\). The maximum degree is \( \Delta(G) = 8 \) and minimum degree is \( \delta(G) = 1 \). The adjacency matrix for the zero-divisor graph of \( R_2 = Z_3 \times Z_6 \) is \( M_2 = \begin{bmatrix} 0 & A_{8 \times 5} \end{bmatrix} \), where \( A_{8 \times 5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \).

\( O_{8 \times 8} \) is the zero matrix and \( A_{13 \times 13}^T \) is the transpose of \( A_{8 \times 5} \).

Properties of adjacency matrix \( M_2 \):

(i) The determinant of the adjacency matrix \( M_2 \) corresponding to \( G = \Gamma(R_2) \) is 0.
(ii) The rank of the adjacency matrix \( M_2 \) corresponding to \( G = \Gamma(R_2) \) is 2.
(iii) The adjacency matrix \( M_2 \) corresponding to \( G = \Gamma(R_2) \) is symmetric and singular.

Case 2: When \( p = 5 \) we have \( R_2 = Z_5 \times Z_{10} \). The ring \( R_2 \) has 33 non-zero zero-divisors. In this case \( V = Z(R_2)^* = \{(1,0),(2,0),(3,0),(4,0),(1,2),(1,4),(1,5),(1,6),(1,8),(2,2),(2,4),(2,5),(2,6),(2,8),(3,2),(3,6),(3,8),(4,2),(4,4),(4,5),(4,6),(4,8),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9)\} \) and the zero-divisor graph \( G = \Gamma(R_2) \) is given by:

![Graph](image)
The closed neighborhoods of the vertices are $N_{Z_2}(1,0) = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (1,0)\}$, $N_{Z_2}(0,0) = \{(1,0), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9)\}$, $N_{Z_2}(0,1) = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (1,0)\}$, $N_{Z_2}(0,2) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,3) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,4) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,5) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,6) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,7) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,8) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$, $N_{Z_2}(0,9) = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$.

The adjacency matrix for the zero-divisor graph of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is

$$A_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \begin{bmatrix} O_{\mathbb{Z}_2 \times \mathbb{Z}_2} & A_{\mathbb{Z}_2 \times \mathbb{Z}_2} \\ A_{\mathbb{Z}_2 \times \mathbb{Z}_2}^T & B_{\mathbb{Z}_2 \times \mathbb{Z}_2} \end{bmatrix} \in \mathbb{Z}_{3 \times 3}$$

and

$$B_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$O_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is the zero matrix and $A_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is the transpose of $A_{\mathbb{Z}_2 \times \mathbb{Z}_2}$.

**Properties of adjacency matrix $M_2$:**

(i) The determinant of the adjacency matrix $M_2$ corresponding to $G = \Gamma(R_2)$ is 0.

(ii) The rank of the adjacency matrix $M_2$ corresponding to $G = \Gamma(R_2)$ is 2.

(iii) The adjacency matrix $M_2$ corresponding to $G = \Gamma(R_2)$ is symmetric and singular.

**Generalization for $R_2 = \mathbb{Z}_p \times \mathbb{Z}_p$ ($p$ is an odd prime number):**

**Lemma 3.1:** The number of vertices of $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is $p$ and $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is $K_{1,p-1}$, where $p$ is an odd prime number.

**Proof:** The multiples of 2 less than $2p$ are $2, 4, 6, \ldots, 2(p-1)$. The non-zero zero-divisors of $\mathbb{Z}_2 \times \mathbb{Z}_p$ are $2, 4, 6, \ldots, 2(p-1)$. If $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is the zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p$, then the vertices of $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ are the non-zero zero-divisors of $\mathbb{Z}_2 \times \mathbb{Z}_p$. So, the vertex set of $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is $\mathcal{Z}(\mathbb{Z}_2 \times \mathbb{Z}_p)$ and $p$ and $2, 4, 6, \ldots, 2(p-1)$ are the vertices of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$. Hence, the number of vertices of $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is $p$.

Also, in $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$, $p$ is adjacent to remaining vertices $2, 4, 6, \ldots, 2(p-1)$. This gives $G = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is $K_{1,p-1}$.

**Theorem 3.2:** Let $R_2$ be a finite commutative ring such that $R_2 = \mathbb{Z}_p \times \mathbb{Z}_p$ ($p$ is an odd prime number). Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $\mathcal{Z}(R_2)$. Then number of vertices of $G = \Gamma(R_2)$ is $p^2 + 2p - 2$, $\Delta(G) = p^2 - 1$ and $\delta(G) = 1$. 

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Proof: Let $R_2$ be a finite commutative ring such that $R_2 = \mathbb{Z}_p \times \mathbb{Z}_{2p}$ ($p$ is an odd prime number). Let $R_2^* = R_2 - \{0\}$. Then $R_2^*$ can be partitioned into disjoint sets $A, B, C, D$ and $E$ such that $A = \{(u, 0) : u \in \mathbb{Z}_p^* \}, B = \{(0, v) : v \in \mathbb{Z}_{2p}^* \}$ and $v \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$, $C = \{(0, w) : w \in \mathbb{Z}_{2p}^* \}$ and $w \in \mathbb{Z} (\mathbb{Z}_{2p})^*$, $D = \{(a, b) : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_{2p}^* \}$ and $a \in \mathbb{Z} (\mathbb{Z}_{2p})^*$ and $E = \{(c, d) : c \in \mathbb{Z}_p^*, d \in \mathbb{Z}_{2p}^* \}$ and $d \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$ respectively. Clearly, all the elements in $A, B, C$ are non-zero zero-divisors.

Let $(a, b) \in D$. Then $(a, b)$ is of the form either $(a, p)$ or $(a, q)$, where $q = 2m, 1 \leq m \leq p - 1$. Again let $(0, w) \in C$. Similarly, $(0, w)$ is of the form either $(0, p)$ or $(0, q)$, where $q = 2m, 1 \leq m \leq p - 1$. Now $p | p$ and $w | w$. This gives $p \mathbb{Z}_p \mathbb{Z}_{2p}$.

Therefore, $(a, p) \in D$ and $(a, q) \notin D$. Hence, every element of $D$ is a non-zero zero-divisor. But product of any two elements of $D$ is not equal to zero. Also, product of any element of $E$ with any element of $A, B, C$ and $D$ is not equal to zero because, $cu \neq 0$ for $c, u \in \mathbb{Z}_p^*, dv \neq 0$ for $d, v \in \mathbb{Z}_{2p}^*$ and $w \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$, $dv \neq 0$ for $d, w \in \mathbb{Z}_{2p}^*$ and $d \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$, $w \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$ and $ca \neq 0$ for $c, a \in \mathbb{Z}_p^*$ respectively. So, no element of $E$ is a non-zero zero-divisor. Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $Z(R_2)$. Then $Z(R_2)$ can be partitioned into four disjoint sets $A, B, C$ and $D$. Now using the Lemma 3.1 we have $|A| = |\mathbb{Z}_p^*| = p - 1$, $|B| = |\mathbb{Z}_{2p}^*| = p - 1$, $|C| = |\mathbb{Z} (\mathbb{Z}_{2p})^*| = p$, $|D| = |\mathbb{Z}_p^*| |\mathbb{Z} (\mathbb{Z}_{2p})^*| = (p - 1) \neq p = p^2 - p$.

Therefore, $|Z(R_2)| = |A| + |B| + |C| + |D| = (p - 1) + (p - 1) + p = p^2 - 2$. So, the number of vertices of $G = \Gamma(R_2)$ is $p^2 + 2p - 2$.

Let $s = (u, 0)$ be any vertex of $A$.

(i) Every vertex of $A$ is adjacent to every vertex of $B$. So, $s$ is adjacent to $p - 1$ vertices of $B$.

(ii) Every vertex of $A$ is adjacent to every vertex of $C$. So, $s$ is adjacent to $p - 1$ vertices of $C$.

(iii) Any vertex of $A$ is not adjacent to any vertex of $D$ as $ua \neq 0$ for $u, a \in \mathbb{Z}_p^*$. Therefore, $deg_G(s) = (p - 1) + p = 2p - 1$.

Let $t = (0, v)$ be any vertex of $B$.

(i) Every vertex of $B$ is adjacent to every vertex of $A$. So, $t$ is adjacent to $p - 1$ vertices of $A$.

(ii) Any vertex of $B$ is not adjacent to any vertex of $C$ as $vw \neq 0$ for and $v, w \in \mathbb{Z}_{2p}^*$ and $v \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$, $w \in \mathbb{Z} (\mathbb{Z}_{2p})^*$.

(iii) Any vertex of $B$ is not adjacent to any vertex of $D$ as $vb \neq 0$ for $v, b \in \mathbb{Z}_{2p}^*$ and $v \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$, $b \in \mathbb{Z} (\mathbb{Z}_{2p})^*$.

Therefore, $deg_G(t) = p - 1$.

Let $x = (0, w)$ be any vertex of $C$. Then either $x = (0, p)$ or $x = (0, q)$, where $q = 2m, 1 \leq m \leq p - 1$.

(i) Every vertex of $C$ is adjacent to every vertex of $A$. So, $x$ is adjacent to $p - 1$ vertices of $A$.

(ii) Case 1: If $x = (0, p)$, then it is adjacent to $p - 1$ vertices of $C$.

Case 2: If $x = (0, q)$, then it is adjacent to only one vertex of $C$.

(iii) Case 1: If $x = (0, p)$, then it is adjacent to $|Z(R_2)|^2 = (p - 1)^2$ vertices of $D$.

Case 2: If $x = (0, q)$, then it is adjacent to $|Z(R_2)| = p - 1$ vertices of $D$.

(iv) Any vertex of $C$ is not adjacent to any vertex of $B$ as $vw \neq 0$ for $w, v \in \mathbb{Z}_{2p}^*$ and $w \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$.

Therefore, $deg_G(x) = (p - 1) + (p - 1) = 2p - 1$. And if $x = (0, q)$, then $deg_G(x) = (p - 1) + 1 + (p - 1) = 2p - 1$.

Let $y = (a, b)$ be any vertex of $D$. Then either $y = (a, p)$ or $y = (a, q)$, where $a \in \mathbb{Z}_p^*$ and $q = 2m, 1 \leq m \leq p - 1$.

(i) Case 1: If $y = (a, p)$, then it is adjacent to $p - 1$ vertices of $C$.

Case 2: If $y = (a, q)$, then it is adjacent to only one vertex of $C$.

(ii) Any vertex of $D$ is not adjacent to any vertex of $A$ as $ua \neq 0$ for $u, a \in \mathbb{Z}_p^*$.

(iii) Any vertex of $D$ is not adjacent to any vertex of $B$ as $bv \neq 0$ for $b, v \in \mathbb{Z}_{2p}^*$ and $b \notin \mathbb{Z} (\mathbb{Z}_{2p})^*$.

Therefore, if $y = (a, p)$, then $deg_G(y) = p - 1$ and if $y = (a, q)$, then $deg_G(y) = 1$.

Hence, we have $\Delta(G) = p^2 - 1$ and $\delta(G) = 1$.

Theorem 3.3: Let $M_2$ be the adjacency matrix for the zero-divisor graph $G = \Gamma(R_2)$ of $R_2 = \mathbb{Z}_p \times \mathbb{Z}_{2p}$ ($p$ is an odd prime number). Then (i) determinant of $M_2$ is zero (ii) $M_2$ is symmetric and singular.

Proof: Follows from Theorem 2.3.

Theorem 3.4: Let $R_2$ be a finite commutative ring such that $R_2 = \mathbb{Z}_p \times \mathbb{Z}_{2p}$ ($p$ is an odd prime number). Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $V = Z(R_2)$. Then $n_d(V) = 2p + \Delta(G) - \delta(G)$, where $n_d(V)$ is the neighborhood number, $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of $G$ respectively.

Proof: Let $R_2$ be a finite commutative ring such that $R_2 = \mathbb{Z}_p \times \mathbb{Z}_{2p}$ ($p$ is an odd prime number). Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $V = Z(R_2)$. Since, $G = \Gamma(R_2)$ is connected [1], we have $n_d(V) = |N_{G} (V)| = \left\| V \right\| \left\| Z(R_2) \right\| ^{C}$.

But from Theorem 3.2, we have $\left\| Z(R_2) \right\| ^{C} = p^2 + 2p - 2$. Therefore, $n_d(V) = p^2 + 2p - 2$. This implies $n_d(V) = 2p + (p^2 - 1) - 1$. Also $\Delta(G) = p^2 - 1$ and $\delta(G) = 1$ [from Theorem 3.2]. This gives $n_d(V) = 2p + \Delta(G) - \delta(G)$.

Remark: If $p = 2$, then $R_2 = Z_2 \times Z_4$. So, this case coincides with the case $I$ of section 2.

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4. CONSTRUCTION OF ZERO-DIVISOR GRAPH FOR $R_3 = \mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$ (FOR THAT ODD PRIME $p$ FOR WHICH $p^2 - 2$ IS A PRIME NUMBER):

Thirdly, we construct the zero-divisor graph for the ring $R_3 = \mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$ (for that odd prime $p$ for which $p^2 - 2$ is a prime number) and analyze the graph. We start with the cases $p = 3$ and $p = 5$ and then generalize the cases.

**Case 1:** When $p = 3$ we have $R_3 = \mathbb{Z}_3 \times \mathbb{Z}_7$.

The ring $R_3$ has 8 non-zero zero-divisors. In this case $V = |Z(R_3)^*| = \{(1,0),(2,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6)\}$ and the zero-divisor graph $\Gamma = \Gamma(R_3)$ is given by:

![Fig: 5](image)

The maximum degree is $\Delta(G) = 6$ and minimum degree is $\delta(G) = 2$. The adjacency matrix for the zero-divisor graph of $R_3 = \mathbb{Z}_3 \times \mathbb{Z}_7$ is $M_3 = \begin{bmatrix} A_{2 \times 6} & O_{2 \times 6} \\ A^T_{6 \times 2} & O_{6 \times 6} \end{bmatrix}_{3 \times 8}$ where all the entries of $A_{2 \times 6}$ is 1, $A^T_{6 \times 2}$ is the transpose of $A_{2 \times 6}$ and $O_{2 \times 6}$, $O_{6 \times 6}$ are the zero matrices.

**Properties of adjacency matrix $M_3$:**

(i) The determinant of the adjacency matrix $M_3$ corresponding to $G = \Gamma(R_3)$ is 0.

(ii) The rank of the adjacency matrix $M_3$ corresponding to $G = \Gamma(R_3)$ is 2.

(iii) The adjacency matrix $M_3$ corresponding to $G = \Gamma(R_3)$ is symmetric and singular.

**Case 2:** When $p = 5$ we have $R_3 = \mathbb{Z}_5 \times \mathbb{Z}_3$.

The ring $R_3$ has 26 non-zero zero-divisors. In this case $V = |Z(R_3)^*| = \{(1,0),(2,0),(3,0),(4,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),(0,10),(0,11),(0,12),(0,13),(0,14),(0,15),(0,16),(0,17),(0,18),(0,19),(0,20),(0,21),(0,22)\}$ and the zero-divisor graph $G = \Gamma(R_3)$ is given by:

![Fig: 6](image)

The closed neighborhoods of the vertices are $N_G([1,0]) = \{(1,0),(2,0),(3,0),(4,0),(0,2),(0,3),(0,4),(0,5),(0,6)\}$ and $N_G([2,0]) = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(2,0),(0,12)\}$. The neighborhood of $V$ is given by $N_G(V) = \{(1,0),(2,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6)\}$.

The zero-divisor graph $\Gamma(G) = \Gamma(R_3)$ is given by:

![Fig: 6](image)

The closed neighborhoods of the vertices are $N_G([1,0]) = \{(1,0),(2,0),(3,0),(4,0),(0,2),(0,3),(0,4),(0,5),(0,6)\}$ and $N_G([2,0]) = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(2,0),(0,12)\}$. The neighborhood of $V$ is given by $N_G(V) = \{(1,0),(2,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6)\}$.
degree is $\Delta(G) = 22$ and minimum degree is $\delta(G) = 4$. The adjacency matrix for the zero-divisor graph of $R_1 = \mathbb{Z}_p \times \mathbb{Z}_p$ is given by

$$M_1 = \begin{bmatrix}
O_{4 \times 4} & A_{4 \times 22} \\
A^T_{22 \times 4} & O_{22 \times 22}
\end{bmatrix}_{26 \times 26},$$

where all the entries of $A_{4 \times 22}$ are 1, $A^T_{22 \times 4}$ is the transpose of $A_{4 \times 22}$ and $O_{4 \times 4}, O_{22 \times 22}$ are the zero matrices.

Properties of adjacency matrix $M_2$:

(i) The determinant of the adjacency matrix $M_1$ corresponding to $G = \Gamma(R)$ is 0.
(ii) The rank of the adjacency matrix $M_1$ corresponding to $G = \Gamma(R)$ is 2.
(iii) The adjacency matrix $M_1$ corresponding to $G = \Gamma(R)$ is symmetric and singular.

Generalization for $R_3 = \mathbb{Z}_p \times \mathbb{Z}_p$ (for odd prime $p$ which $p^2-2$ is a prime number):

Theorem 4.1: Let $R_3$ be a finite commutative ring such that $R_3 = \mathbb{Z}_p \times \mathbb{Z}_p$ (for odd prime $p$ for which $p^2-2$ is a prime number). Let $G = \Gamma(R_3)$ be the zero-divisor graph with vertex set $Z(R_3)*$. Then number of vertices of $G = \Gamma(R_3)$ is $p^2 + p - 4$, $\Delta(G) = p^2 - 3$ and $\delta(G) = p - 1$.

Proof: Let $R_3$ be a finite commutative ring such that $R_3 = \mathbb{Z}_p \times \mathbb{Z}_p$ (for odd prime $p$ for which $p^2-2$ is a prime number). Let $R_3^* = R_3 - \{0\}$. Then $R_3^*$ can be partitioned into disjoint sets $A, B \subset \mathbb{Z}_p^*$, where $A = \{(u, 0): u \in \mathbb{Z}_p^*\}$, $B = \{(0, v): v \in \mathbb{Z}_p^* \}$. Clearly, all the elements of $A$ and $B$ are non-zero divisors. But product of any two elements of $C$ is not equal to zero. Also, product of any element of $C$ with any element of $A$ and $B$ is not equal to zero because, $au \not= 0$ for $a \in \mathbb{Z}_p^*$, $bv \not= 0$ for $b \in \mathbb{Z}_p^*$, $u \not= 0$ for $u \in \mathbb{Z}_p^*$. So, no element of $C$ is a non-zero divisor. Let $G = \Gamma(R_3)$ be the zero-divisor graph with vertex set $Z(R_3)*$. Then $Z(R_3)*$ can be partitioned into two disjoint sets $A$ and $B$. Now, $|A| = |Z_p^*| = p - 1$ and $|B| = |Z_p^*| = p - 3$. Therefore, $|Z(R_3)*| = |A| + |B| = |Z_p^*| + |Z_p^*| = (p - 1) + (p - 3) = p^2 - 4$. So, the number of vertices of $G = \Gamma(R_3)$ is $p^2 + p - 4$.

5. DEFINITIONS AND RELATIONS:

Let $R$ be a commutative ring with unity and let $a \in R$.

Then annihilator of $a$ is denoted by $ann(a)$ and defined by $ann(a) = \{x \in R : ax = 0\}$. Let $ann^*(a) = \{x \not= 0 \in R : R \not= 0\}$.

The degree of a vertex $v$ of a graph $G$ denoted by $deg(v)$ is the number of lines incident with $v$.

Given a zero-divisor graph $\Gamma(R)$ with vertex set $Z(R)*$, then degree of a vertex $v$ of $\Gamma(R)$ is given by $deg(v) = |ann(v^*)|$. Let $A$ and $B$ be two commutative rings with unity. Then the direct product $A \times B$ of $A$ and $B$ is also a commutative ring with unity.

Let $G$ be a graph and $V(G)$ be the vertex set of $G$. Let $a, b \in V(G)$. We define a relation $\equiv$ on $V(G)$ as follows. For $a, b \in V(G)$, $a$ is related to $b$ under the relation $\equiv$ if and only if $a$ and $b$ are not adjacent and for any $x \in V(G)$, $a$ and $x$ are adjacent if and only if $b$ and $x$ are adjacent. We denote this relation by $a \equiv b$.

6. Results of annihilators on $\Gamma(A \times B)$:

Theorem 6.1: The relation $\equiv$ is an equivalence relation on $V(G)$, where $G$ is any graph.

Proof: For every $a \in V(G)$, we have $a \equiv a$, as $G$ has no self-loop. For $a, b \in V(G), a \equiv b$, then clearly, $b \equiv a$. Again let
If possible suppose, \( a \) and \( c \) are adjacent. Then we have \( b \) and \( c \) are also adjacent, a contradiction. So, \( a \) and \( c \) are not adjacent. Also for \( x \in V(G) \), \( a \) and \( x \) are adjacent \( \iff b \) and \( x \) are adjacent \( \iff c \) and \( x \) are adjacent. Therefore, \( a \not\sim c \). Hence, the relation \( \not\sim \) is an equivalence relation on \( V(G) \).

**Theorem 6.2:** For distinct \( a, b \in \mathbb{Z}(A \times B) \), \( a \not\sim b \) in \( \Gamma(A \times B) \) if and only if \( \text{ann}(a) = \{a\} \neq \text{ann}(b) \). Moreover, if \( a \not\sim b \) in \( \Gamma(R \times R) \), then \( \text{ann}(a) = \{a\} = \text{ann}(b) = \{b\} \) and \( \text{ann}(a) - \{a\} = \text{ann}(b) - \{b\} \).

**Proof:** First suppose, for distinct \( a, b \in \mathbb{Z}(A \times B) \), \( a \not\sim b \) in \( \Gamma(A \times B) \). Let \( x \in \text{ann}(a) - \{a\} \). This gives \( ax = 0, a \neq x \). So, \( a \) and \( x \) are adjacent. Since \( a \not\sim x \) we have \( b \not\sim x \) and \( a \not\sim x \). Therefore, we have \( bx = 0, b \neq x \). Hence, \( x \in \text{ann}(b) \). This implies \( \text{ann}(a) - \{a\} \neq \text{ann}(b) - \{b\} \). Similarly, \( \text{ann}(b) - \{b\} \neq \text{ann}(a) - \{a\} \). This gives \( \text{ann}(a) - \{a\} = \text{ann}(b) - \{b\} \).

Conversely suppose, \( \text{ann}(a) - \{a\} = \text{ann}(b) - \{b\} \). Assume that \( a \) and \( b \) are adjacent. This gives \( ab = 0 \iff a \not\sim b \). Hence, \( x \in \text{ann}(a) - \{a\} \). Then \( ax = 0, a \neq x \). Therefore, \( a \) and \( x \) are adjacent. Since \( a \not\sim x \) we have \( b \not\sim x \) and \( a \not\sim x \). Therefore, we have \( bx = 0, b \neq x \). Hence, \( x \in \text{ann}(b) \). This implies \( \text{ann}(a) - \{a\} = \text{ann}(b) - \{b\} \).

Example 6.3: Consider the commutative ring \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with \( \{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}, \{1,2\}, \{1,3\} \) and zero-divisor graph \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2) \). Here \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) is \( \{0,1\}, \{0,2\}, \{0,3\}, \{1,0\}, \{1,1\}, \{1,2\}\). The possible edges are \( \{0, 1\}, \{1,0\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,0\}\) and \( \{0,2\}, \{1,2\}\). The pairs \( \{0, 1\}, \{0,2\}, \{0,3\} \) and \( \{1,0\}, \{1,2\}\) establish the existence of relation \( \not\sim \) and Theorem 6.1.

7. CONCLUSIONS:

In this paper, we study the adjacency matrix and neighborhood associated with zero-divisor graph for direct product of finite commutative rings. Neighborhoods may be used to represent graphs in computer algorithms, via the adjacency list and adjacency matrix representations. Neighborhoods are also used in the clustering coefficient of a graph, which is a measure of the average density of its neighborhoods. In addition, many important classes of graphs may be defined by properties of their neighborhoods.

8. REFERENCES:
