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On the Adjacency Matrix and Neighborhood Associated with Zero-divisor Graph for Direct Product of Finite Commutative Rings

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Abstract: The main purpose of this paper is to study the zero-divisor graph for direct product of finite commutative rings. In our present investigation we discuss the zero-divisor graphs for the following direct products: direct product of the ring of integers under addition and multiplication modulo p and the ring of integers under addition and multiplication modulo p^2 for a prime number p, direct product of the ring of integers under addition and multiplication modulo p and the ring of integers under addition and multiplication modulo $2p$ for an odd prime number p and direct product of the ring of integers under addition and multiplication modulo p and the ring of integers under addition and multiplication modulo $p^2 - 2$ for that odd prime p for which $p^2 - 2$ is a prime number. The aim of this paper is to give some new ideas about the neighborhood, the neighborhood number and the adjacency matrix corresponding to zero-divisor graphs for the above mentioned direct products. Finally, we prove some results of annihilators on zero-divisor graph for direct product of A and B for any two commutative rings A and B with unity

Keywords: Zero-divisor, Commutative ring, Adjacency matrix, Neighborhood, Zero-divisor graph, Annihilator.

AMS Classification (2010): 05Cxx; 05C25; 05C50.

1. INTRODUCTION

The idea of zero-divisor graph of a commutative ring was first introduced by I. Beck [2] in 1988. D. F. Anderson and P.S. Livinsgston [1] redefined the concept of zero-divisor graph in 1999. F. R. DeMeyer, T. Mckenzie and K. Schneider [3] extended the concept of zero-divisor graph for commutative semi-group in 2002. The notion of zero-divisor graph had been extended for non-commutative rings by S. P. Redmond [9] in 2002. Recently, P. Sharma, A. Sharma and R. K. Vats [10] have discussed the neighborhood set, the neighborhood number and the adjacency matrix of zero-divisor graphs for the rings $Z_p \times Z_p$ and $Z_p[i] \times Z_p[i]$, where p is a prime number.

In this paper R_1 denotes the finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ (p is a prime number), R_2 denotes the finite commutative ring such that $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number) and R_3 denotes the finite commutative ring such that $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2 - 2$ is a prime number). Let R be a commutative ring with unity and Z(R) be the set of zero-divisors of R; that is Z(R) = $\{x \in R\colon xy = 0$ or $yx = 0$ for some $y \in R^* = R - \{0\}\}$.Then *zero-divisor graph* of R is an *undirected graph* $\Gamma(R)$ with vertex set Z(R)* = Z(R) − {0} such that distinct vertices x and y of Z(R)* are adjacent if and only if $xy = 0$. The neighborhood (or open neighborhood) $N_G(v)$ of a vertex v of a graph G is the set of vertices adjacent to v. The closed neighborhood $N_G[v]$ of a vertex v is the set $N_G(v) \cup \{v\}$. For a set S of vertices, the neighborhood of S is the union of the neighborhoods of the vertices and so it is the set of all vertices adjacent to at least one member of S. For a graph G with vertex set V, the union of the neighborhoods of all the vertices is neighborhood of V and it is denoted by $N_G(V)$. The neighborhood number $n_G(V)$ is the cardinality of $N_G(V)$. If the graph G with vertex set V is connected, then $N_G(V)$ is the vertex set V and the cardinality of $N_G(V)$ is equal to the cardinality of V. If $\Gamma(R)$ is the zero-divisor graph of a commutative ring R with vertex set Z(R)* and since zero-divisor graph is always connected [1], we have $N_{\Gamma(R)}(Z(R)^*) = Z(R)^*$ and $|N_{\Gamma(R)}(Z(R)^*)| = |Z(R)^*|$. Throughout this paper $\Delta(G)$ denotes the maximum degree of a graph G and $\delta(G)$ denotes the minimum degree of a graph G. The adjacency matrix corresponding to zero-divisor graph G is defined as $A = [a_{ij}]$, where $a_{ij} = 1$, if $v_i v_j = 0$ for any vertex v_i and v_j of G and $a_{ij} = 0$, otherwise.

In this paper, we construct zero-divisor graphs for the rings R_1, R_2 and R_3. We obtain the neighborhood and the adjacency matrices corresponding to zero-divisor graphs of R_1, R_2 and R_3. Some properties of adjacency matrices are also obtained. We prove some theorems related to neighborhood and adjacency matrices corresponding to zero-divisor graphs of R_1, R_2 and R_3. Finally, we prove some results of annihilators on zero-divisor graph of $A \times B$, for any two commutative rings A and B with unity.

2. CONSTRUCTION OF ZERO -DIVISOR GRAPH FOR $R_1 = Z_p \times Z_{p^2}$ (*p* IS A PRIME NUMBER):

First, we construct the zero-divisor graph for the ring $R_1 = Z_p \times Z_{p^2}$ (*p* is a prime number) and analyze the graph. We start with the cases $p = 2$ and $p = 3$ and then generalize the cases.

Case1: When $p = 2$ we have $R_1 = Z_2 \times Z_4$.
The ring R_1 has 5 non-zero zero-divisors. In this case $V = Z(R_1)^* = \{(1,0), (0,1), (0,2), (0,3), (1,2)\}$ and the zero-divisor graph $G = \Gamma(R_1)$ is given by:

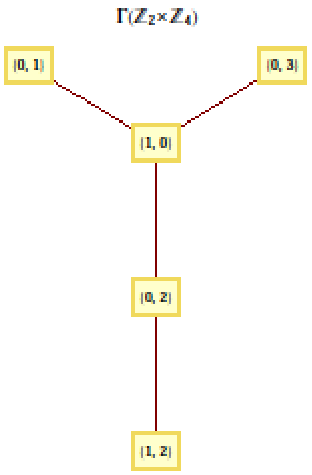

Fig: 1

The closed neighborhoods of the vertices are $N_G[(1,0)] = \{(1,0), (0,1), (0,2),(0,3)\}$, $N_G[(0,1)] = \{(1,0),(0,1)\}$, $N_G[(0,2)] = \{(1,0), (1,2), (0,2)\}$, $N_G[(0,3)] = \{(1,0),(0,3)\}$ and $N_G[(1,2)] = \{(0,2),(1,2)\}$. The neighborhood of *V* is given by $N_G(V) = \{(1,0), (0,1), (0,2), (0,3), (1,2)\}$. The maximum degree is $\Delta(G) = 3$ and minimum degree is $\delta(G) = 1$. The adjacency matrix for the zero-divisor graph of $R_1 = Z_2 \times Z_4$ is

$$M_1 = \begin{bmatrix} 0 & A_{1\times 3} & 0 \\ A^T{}_{3\times 1} & O_{3\times 3} & B_{3\times 1} \\ 0 & B^T{}_{1\times 3} & 0 \end{bmatrix}_{5\times 5} \quad \text{where, } A_{1\times 3} = [1\ 1\ 1],$$

$B_{3\times 1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A^T{}_{3\times 1}$ is the transpose of $A_{1\times 3}$, $B^T{}_{1\times 3}$ is the transpose of $B_{3\times 1}$ and $O_{3\times 3}$ is the zero matrix.

Properties of adjacency matrix M_1:

(i) The determinant of the adjacency matrix M_1 corresponding to $G = \Gamma(R_1)$ is 0.
(ii) The rank of the adjacency matrix M_1 corresponding to $G = \Gamma(R_1)$ is 2.
(iii) The adjacency matrix M_1 corresponding to $G = \Gamma(R_1)$ is symmetric and singular.

Case2: When $p = 3$ we have $R_1 = Z_3 \times Z_9$.
The ring R_1 has 14 non-zero zero-divisors. In this case $V = Z(R_1)^* = \{(1,0), (2,0), (1,3), (1,6), (2,3), (2,6), (0,1), (0,2), (0,3), (0,4), (0,5),(0,6),(0,7),(0,8)\}$ and the zero-divisor graph $G = \Gamma(R_1)$ is given by:

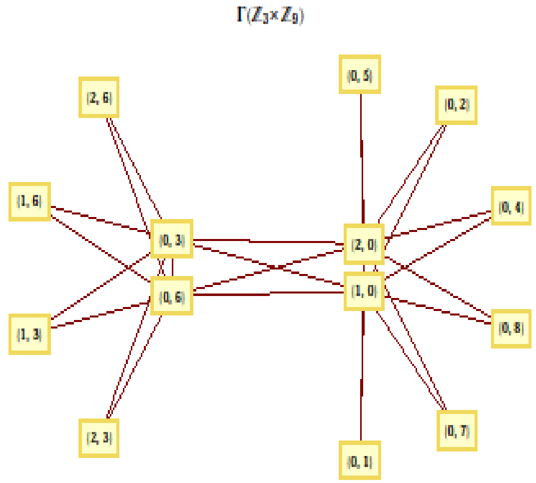

Fig: 2

The closed neighborhoods of the vertices are $N_G[(1,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (1,0)\}$, $N_G[(2,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8),(2,0)\}$, $N_G[(1,3)] = \{(0,3),(0,6),(1,3)\}$, $N_G[(1,6)] = \{(0,3), (0,6),(1,6)\}$, $N_G[(2,3)] = \{(0,3),(0,6),(2,3)\}$, $N_G[(2,6)] = \{(0,3), (0,6),(2,6)\}$, $N_G[(0,1)] = \{(1,0),(2,0),(0,1)\}$, $N_G[(0,2)] = \{(1,0), (2,0),(0,2)\}$, $N_G[(0,3)] = \{(1,0), (2,0), (1,3), (1,6), (2,3), (2,6), (0,6),(0,3)\}$, $N_G[(0,4)] = \{(1,0),(2,0),(0,4)\}$, $N_G[(0,5)] = \{(1,0), (2,0),(0,5)\}$, $N_G[(0,6)] = \{(1,0), (2,0), (1,3), (1,6), (2,3), (2,6), (0,3),(0,6)\}$, $N_G[(0,7)] = \{(1,0), (2,0), (0,7)\}$, $N_G[(0,8)] = \{(1,0), (2,0), (0,8)\}$. The neighborhood of *V* is given by $N_G(V) = \{(1,0), (0,2), (1,3), (1,6), (2,3), (2,6), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8)\}$. The maximum degree is $\Delta(G) = 8$ and minimum degree is $\delta(G) = 2$. The adjacency matrix for the zero-divisor graph of $R_1 = Z_3 \times Z_9$ is $M_1 =$

$$\begin{bmatrix} O_{6\times 6} & A_{6\times 5} & B_{6\times 3} \\ A^T{}_{5\times 6} & O_{5\times 5} & C_{5\times 3} \\ B^T{}_{3\times 6} & C^T{}_{3\times 5} & O_{3\times 3} \end{bmatrix}_{14\times 14} \quad \text{where } A_{6\times 5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$B_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $C_{5\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_{6\times 6}$, $O_{5\times 5}$, $O_{3\times 3}$ are the zero matrices and $A^T{}_{5\times 6}$, $B^T{}_{3\times 6}$, $C^T{}_{3\times 5}$ are the transposes of $A_{6\times 5}$, $B_{6\times 3}$, $C_{5\times 3}$ respectively.

Properties of adjacency matrix M_1:

(i) The determinant of the adjacency matrix M_1 corresponding to $G = \Gamma(R_1)$ is 0.
(ii) The rank of the adjacency matrix M_1 corresponding to $G = \Gamma(R_1)$ is 2.
(iii) The adjacency matrix M_1 corresponding to $G = \Gamma(R_1)$ is symmetric and singular.

Generalization for $R_1 = Z_p \times Z_{p^2}$ (p is a prime number):

Lemma 2.1: The number of vertices of $G = \Gamma(Z_{p^2})$ *is* $p - 1$ and $G = \Gamma(Z_{p^2})$ *is* K_{p-1}, where p is a prime number [4]

Proof: The multiples of p less than p^2 are $p, 2p, 3p, \ldots\ldots, (p-1)p$. These multiples of p are the only non-zero zero-divisors of Z_{p^2}. If $G = \Gamma(Z_{p^2})$ is the zero-divisor graph of Z_{p^2}, then the vertices of $G = \Gamma(Z_{p^2})$ are the non-zero zero-divisors of Z_{p^2}. So, the vertex set of $G = \Gamma(Z_{p^2})$ is $Z(Z_{p^2})^*$ and $p, 2p, 3p, \ldots\ldots, (p-1)p$ are the vertices of $G = \Gamma(Z_{p^2})$. Hence, the number of vertices of $G = \Gamma(Z_{p^2})$ is $p - 1$. Also, in $G = \Gamma(Z_{p^2})$, every vertex is adjacent to every other vertex. This gives $G = \Gamma(Z_{p^2})$ *is* K_{p-1}.

Theorem 2.2: Let R_1 be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ (p is a prime number). Let $G = \Gamma(R_1)$ be the zero-divisor graph with vertex set $Z(R_1)^*$. Then number of vertices of $G = \Gamma(R_1)$ is $2p^2 - p - 1$, $\Delta(G) = p^2 - 1$ and $\delta(G) = p - 1$.

Proof: Let R_1 be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ (p is a prime number). Let $R_1^* = R_1 - \{0\}$. Then R_1^* can be partitioned into disjoint sets A, B, C, D and E such that $A = \{(u, 0) : u \in Z_p^*\}$, $B = \{(0, v) : v \in Z_{p^2}^*$ and $v \notin Z(Z_{p^2})^*\}$, $C = \{(0, w) : w \in Z_{p^2}^*$ and $w \in Z(Z_{p^2})^*\}$, $D = \{(a, b) : a \in Z_p^*, b \in Z_{p^2}^*$ and $b \in Z(Z_{p^2})^*\}$ and $E = \{(c, d) : c \in Z_p^*, d \in Z_{p^2}^*$ and $d \notin Z(Z_{p^2})^*\}$. Clearly, all the elements in A, B, C are non-zero zero-divisors. Let $(a, b) \in D$ and $(0, w) \in C$. Here $b, w \in Z(Z_{p^2})^*$. So, p/b and p/w. This gives p^2/bw. Therefore, $(a, b)\,(0, w) = (0, 0)$. Hence, every element of D is a non-zero zero-divisor. But product of any two elements of E is not equal to zero. Also, product of any element of E with any element of A, B, C and D is not equal to zero because, $cu \neq 0$ for $c, u \in Z_p^*$, $dv \neq 0$ for $d, v \in Z_{p^2}^*$ and $d, v \notin Z(Z_{p^2})^*$, $dw \neq 0$ for $d, w \in Z_{p^2}^*$ and $d \notin Z(Z_{p^2})^*$, $w \in Z(Z_{p^2})^*$ and $ca \neq 0$ for $c, a \in Z_p^*$ respectively. So, no element of E is a non-zero zero-divisor. Let $G = \Gamma(R_1)$ be the zero-divisor graph with vertex set $Z(R_1)^*$.Then $Z(R_1)^*$ can be partitioned into four disjoint sets A, B, C and D. Now using the **Lemma 2.1** we have $|A| = |Z_p^*| = p - 1$, $|B| = |Z_{p^2}^*| - |Z(Z_{p^2})^*| = (p^2-1) - (p-1) = p^2 - 1 - p + 1 = p^2 - p$, $|C| = |Z(Z_{p^2})^*| = p - 1$, $|D| = |Z_p^*|\,|Z(Z_{p^2})^*| = (p-1)(p-1) = p^2 - 2p + 1$.
Therefore, $|Z(R_1)^*| = |A| + |B| + |C| + |D| = (p-1) + (p^2 - p) + (p-1) + (p^2 - 2p + 1) = 2p^2 - p - 1$.

So, the number of vertices of $G = \Gamma(R_1)$ is $2p^2 - p - 1$.

Let $s = (u, 0)$ be any vertex of A.
(i) Every vertex of A is adjacent to every vertex of B. So, s is adjacent to $p^2 - p$ vertices of B.
(ii) Every vertex of A is adjacent to every vertex of C. So, s is adjacent to $p - 1$ vertices of C.
(iii) Any vertex of A is not adjacent to any vertex of D as $ua \neq 0$ for $u, a \in Z_p^*$.
Therefore, $deg_G(s) = (p^2 - p) + (p - 1) = p^2 - 1$.

Let $t = (0, v)$ be any vertex of B.
(i) Every vertex of B is adjacent to every vertex of A. So, t is adjacent to $p - 1$ vertices of A.
(ii) Any vertex of B is not adjacent to any vertex of C as $vw \neq 0$ for $v, w \in Z_{p^2}^*$ and $v \notin Z(Z_{p^2})^*$, $w \in Z(Z_{p^2})^*$.
(iii) Any vertex of B is not adjacent to any vertex of D as $vb \neq 0$ for $v, b \in Z_{p^2}^*$ and $v \notin Z(Z_{p^2})^*$, $b \in Z(Z_{p^2})^*$.
Therefore, $deg_G(t) = p - 1$.

Let $x = (0, w)$ be any vertex of C.
(i) Every vertex of C is adjacent to every vertex of A. So, x is adjacent to $p - 1$ vertices of A.
(ii) Any two vertices of C are adjacent to each other. So, x is adjacent to $p - 2$ vertices of C.
(iii) Every vertex of C is adjacent to every vertex of D. So, x is adjacent to $p^2 - 2p + 1$ vertices of D.
(iv) Any vertex of C is not adjacent to any vertex of B as $wv \neq 0$ for $w, v \in Z_{p^2}^*$ and $w \in Z(Z_{p^2})^*$, $v \notin Z(Z_{p^2})^*$.
Therefore, $deg_G(x) = (p - 1) + (p - 2) + (p^2 - 2p + 1) = p^2 - 2$.

Let $y = (a, b)$ be any vertex of D.
(i) Every vertex of D is adjacent to every vertex of C. So, y is adjacent to $p - 1$ vertices of C.
(ii) Any vertex of D is not adjacent to any vertex of A as $au \neq 0$ for $u, a \in Z_p^*$.
(iii) Any vertex of D is not adjacent to any vertex of B as $bv \neq 0$ for $b, v \in Z_{p^2}^*$ and $b \in Z(Z_{p^2})^*$, $v \notin Z(Z_{p^2})^*$.
Therefore, $deg_G(y) = p - 1$.

Hence, we have $\Delta(G) = p^2 - 1$ and $\delta(G) = p - 1$.
Theorem 2.3: Let M_1 be of the adjacency matrix for the zero-divisor graph $G = \Gamma(R_1)$ of $R_1 = Z_p \times Z_{p^2}$ (p is a prime number). Then (i) determinant of M_1 is zero *(ii)* M_1 is symmetric and singular.

Proof: Let R_1 be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ (p is a prime number). Let $G = \Gamma(R_1)$ be the zero-divisor graph with vertex set $V = Z(R_1)^*$ and M_1 be the adjacency matrix for the zero-divisor graph of $R_1 = Z_p \times Z_{p^2}$.

(i) Since, at least two vertices of $G = \Gamma(R_1)$ are adjacent to same vertex of G, so M_1 contains at least two identical rows (eg. for $Z_2 \times Z_4$). Therefore, the determinant of the adjacency matrix M_1 is zero.

(ii) Clearly M_1 is symmetric. Since, the determinant of the adjacency matrix M_1 is zero, M_1 is singular.

Theorem 2.4: Let R_1 be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ (p is a prime number). Let $G = \Gamma(R_1)$ be the zero-divisor graph with vertex set $V = Z(R_1)^*$.Then $n_G(V) = 2\Delta(G) - \delta(G)$, where $n_G(V)$ is the neighborhood number, $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G respectively.

Proof: Let R_1 be a finite commutative ring such that $R_1 = Z_p \times Z_{p^2}$ (p is a prime number). Let $G = \Gamma(R_1)$ be the zero-divisor graph with vertex set $V = Z(R_1)^*$. Since, $G = \Gamma(R_1)$ is connected [1], we have $n_G(V) = |N_G(V)| = |V| = |Z(R_1)^*|$.

But from **Theorem 2.2**, we have $|Z(R_1)^*| = 2p^2 - p - 1$. Therefore, $n_G(V) = 2p^2 - p - 1$. This implies $n_G(V) = 2(p^2 - 1) - (p - 1)$. Also, $\Delta(G) = p^2 - 1$ and $\delta(G) = p - 1$ [from **Theorem 2.2**]. This gives $n_G(V) = 2\Delta(G) - \delta(G)$.

3. CONSTRUCTION OF ZERO -DIVISOR GRAPH FOR $R_2 = Z_p \times Z_{2p}$ (p IS AN ODD PRIME NUMBER):

Secondly, we construct the zero-divisor graph for the ring $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number) and analyze the graph. We start with the cases $p = 3$ and $p = 5$ and then generalize the cases.

Case1: When $p = 3$ we have $R_2 = Z_3 \times Z_6$.
The ring R_2 has 13 non-zero zero-divisors. In this case $V = Z(R_2)^* = \{(1,0),(2,0),(1,2),(1,3), (1,4), (2,2), (2,3), (2,4), (0,1), (0,2),(0,3),(0,4),(0,5)\}$ and the zero-divisor graph $G = \Gamma(R_2)$ is given by:

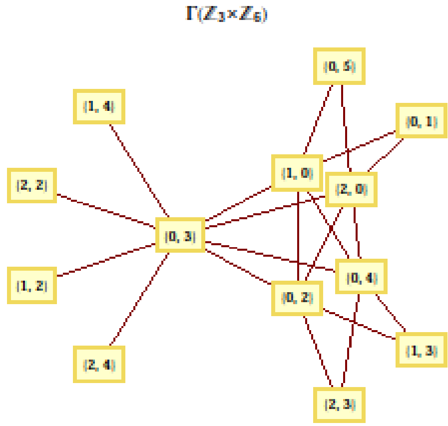

Fig: 3

The closed neighborhoods of the vertices are $N_G[(1,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (1,0)\}$, $N_G[(2,0)] = \{(0,1), (0,2), (0,3), (0,4),(0,5),(2,0)\}$, $N_G[(1,2)] = \{(0,3),(1,2)\}$, $N_G[(1,3)] = \{(0,2), (0,4), (1,3)\}$, $N_G[(1,4)] = \{(0,3), (1,4)\}$, $N_G[(2,2)] = \{(0,3), (2,2)\}$, $N_G[(2,3)] = \{(0,2),(0,4),(2,3)\}$, $N_G[(2,4)] = \{(0,3), (2,4)\}$, $N_G[(0,1)] = \{(1,0), (2,0), (0,1)\}$, $N_G[(0,2)] = \{(1,0), (2,0), (1,3), (2,3), (0,3), (0,2)\}$, $N_G[(0,3)] = \{(1,0), (2,0), (1,2),(1,4),(2,2),(2,4),(0,2),(0,4),(0,3)\}$, $N_G[(0,4)] = \{(1,0),(2,0),(1,3),(2,3),(0,3),(0,4)\}$, $N_G[(0,5)] = \{(1,0), (2,0), (0,5)\}$.The neighborhood of V is given by $N_G(V) = \{(1,0), (2,0), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (0,1), (0,2), (0,3), (0,4), (0,5)\}$.The maximum degree is $\Delta(G) = 8$ and minimum degree is $\delta(G) = 1$. The adjacency matrix for the zero-divisor graph of $R_2 = Z_3 \times Z_6$ is $M_2 = \begin{bmatrix} O_{8\times 8} & A_{8\times 5} \\ A^T{}_{5\times 8} & B_{5\times 5} \end{bmatrix}_{13\times 13}$,

where $A_{8\times 5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $B_{5\times 5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

$O_{8\times 8}$ is the zero matrix and $A^T{}_{5\times 8}$ is the transpose of $A_{8\times 5}$.

Properties of adjacency matrix M_2:

(i) The determinant of the adjacency matrix M_2 corresponding to $G = \Gamma(R_2)$ is 0.
(ii) The rank of the adjacency matrix M_2 corresponding to $G = \Gamma(R_2)$ is 2.
(iii) The adjacency matrix M_2 corresponding to $G = \Gamma(R_2)$ is symmetric and singular.

Case2: When $p = 5$ we have $R_2 = Z_5 \times Z_{10}$.
The ring R_2 has 33 non-zero zero-divisors. In this case $V = Z(R_2)^* = \{(1,0),(2,0),(3,0),(4,0), (1,2), (1,4), (1,5), (1,6), (1,8), (2,2),(2,4),(2,5),(2,6),(2,8),(3,2),(3,4), (3,5), (3,6), (3,8), (4,2), (4,4), (4,5), (4,6), (4,8), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)\}$ and the zero-divisor graph $G = \Gamma(R_2)$ is given by:

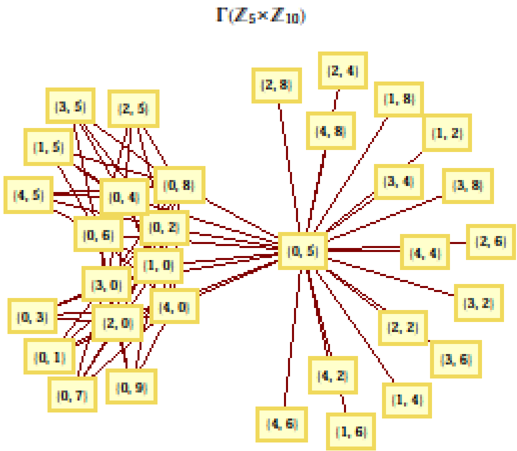

Fig: 4

The closed neighborhoods of the vertices are $N_G[(1,0)]$ = {(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (1,0)}, $N_G[(2,0)]$ = {(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (2,0)}, $N_G[(3,0)]$ = {(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (3,0)}, $N_G[(4,0)]$ = {(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (4,0)}, $N_G[(1,2)]$ = {(0,5), (1,2)}, $N_G[(1,4)]$ = {(0,5), (1,4)}, $N_G[(1,5)]$ = {(0,2), (0,4), (0,6), (0,8), (1,5)}, $N_G[(1,6)]$ = {(0,5), (1,6)}, $N_G[(1,8)]$ = {(0,5),(1,8)}, $N_G[(2,2)]$ = {(0,5),(2,2)}, $N_G[(2,4)]$ = {(0,5), (2,4)}}, $N_G[(2,5)]$ = {(0,2), (0,4), (0,6), (0,8), (2,5)}, $N_G[(2,6)]$ = {(0,5), (2,6)}, $N_G[(2,8)]$ = {(0,5),(2,8)}, $N_G[(3,2)]$ = {(0,5), (3,2)}, $N_G[(3,4)]$ = {(0,5), (3,4)}, $N_G[(3,5)]$ = {(0,2), (0,4), (0,6),(0,8),(3,5)}, $N_G[(3,6)]$ ={(0,5), (3,6)}, $N_G[(3,8)]$ = {(0,5), (3,8)}, $N_G[(4,2)]$ = {(0,5), (4,2)}}, $N_G[(4,4)]$ = {(0,5), (4,4)}, $N_G[(4,5)]$ = {(0,2), (0,4), (0,6), (0,8), (4,5)}, $N_G[(4,6)]$ = {(0,5), (4,6)}, $N_G[(4,8)]$ = {(0,5), (4,8)}, $N_G[(0,1)]$ = {(1,0), (2,0), (3,0), (4,0), (0,1)}, $N_G[(0,2)]$ = {(1,0), (2,0), (3,0), (4,0), (0,5), (1,5), (2,5), (3,5), (4,5), (0,2)}, $N_G[(0,3)]$ = {(1,0), (2,0), (3,0), (4,0),(0,3)}, $N_G[(0,4)]$ = {(1,0), (2,0), (3,0), (4,0), (0,5), (1,5), (2,5), (3,5), (4,5), (0,4)}, $N_G[(0,5)]$ = {(1,0), (2,0),(3,0), (4,0), (1,2), (1,4),(1,6), (1,8), (2,2), (2,4), (2,6), (2,8), (3,2), (3,4), (3,6), (3,8), (4,2), (4,4), (4,6), (4,8), (0,2), (0,4), (0,6), (0,8), (0,5)}, $N_G[(0,6)]$ = {(1,0),(2,0),(3,0), (4,0), (1,5),(2,5), (3,5), (4,5), (0,5),(0,6)}, $N_G[(0,7)]$ = {(1,0), (2,0), (3,0), (4,0), (0,7)}, $N_G[(0,8)]$ = {(1,0), (2,0), (3,0), (4,0), (1,5), (2,5), (3,5), (4,5), (0,5),(0,8)}, $N_G[(0,9)]$ = {(1,0), (2,0), (3,0), (4,0), (0,9)}.The neighborhood of V is given by $N_G(V)$ = {(1,0), (2,0), (3,0), (4,0), (1,2), (1,4), (1,5), (1,6), (1,8), (2,2), (2,4), (2,5), (2,6), (2,8), (3,2), (3,4),(3,5),(3,6),(3,8), (4,2), (4,4), (4,5),(4,6), (4,8), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9)}. The maximum degree is $\Delta(G) = 24$ and minimum degree is $\delta(G) = 1$. The adjacency matrix for the zero-divisor graph of

$$R_2 = Z_5{\times}Z_{10} \text{ is } M_2 = \begin{bmatrix} O_{24\times 24} & A_{24\times 9} \\ A^T{}_{9\times 24} & B_{9\times 9} \end{bmatrix}_{33\times 33},$$

$$A_{24\times 9} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \end{bmatrix},$$

$$B_{9\times 9} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} \quad \text{and}$$

$O_{24\times 24}$ is the zero matrix and $A^T{}_{9\times 24}$ is the transpose of $A_{24\times 9}$.

Properties of adjacency matrix M_2:

(i) The determinant of the adjacency matrix M_2 corresponding to $G = \Gamma(R_2)$ is 0.
(ii) The rank of the adjacency matrix M_2 corresponding to $G = \Gamma(R_2)$ is 2.
(iii) The adjacency matrix M_2 corresponding to $G = \Gamma(R_2)$ is symmetric and singular.

Generalization for $R_2 = Z_p{\times}Z_{2p}$ (p is an odd prime number):

Lemma 3.1: The number of vertices of $G = \Gamma(Z_{2p})$ is p and $G = \Gamma(Z_{2p})$ is $K_{1,\,p-1}$, where p is an odd prime number.

Proof: The multiples of 2 less than $2p$ are 2, 4, 6, ……, $2(p-1)$. The non-zero zero-divisors of Z_{2p} are p and 2, 4, 6, ……,$2(p-1)$. If $G = \Gamma(Z_{2p})$ is the zero-divisor graph of Z_{2p}, then the vertices of $G = \Gamma(Z_{2p})$ are the non-zero zero-divisors of Z_{2p}. So, the vertex set of $G = \Gamma(Z_{2p})$ is $Z(Z_{2p})^*$ and p and 2, 4, 6, ……,$2(p-1)$ are the vertices of $\Gamma(Z_{2p})$. Hence, the number of vertices of $G = \Gamma(Z_{2p})$ is p. Also, in $G = \Gamma(Z_{2p})$, p is adjacent to remaining vertices 2, 4, 6, ……,$2(p-1)$. This gives $G = \Gamma(Z_{2p})$ is $K_{1,\,p-1}$.

Theorem 3.2: Let R_2 be a finite commutative ring such that $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number). Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $Z(R_2)^*$. Then number of vertices of $G = \Gamma(R_2)$ is p^2+2p-2, $\Delta(G) = p^2-1$ and $\delta(G) = 1$.

Proof: Let R_2 be a finite commutative ring such that that $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number). Let $R_2^* = R_2 - \{0\}$. Then R_2^* can be partitioned into disjoint sets A, B, C, D and E such that $A = \{(u, 0) : u \in Z_p^*\}$, $B = \{(0, v) : v \in Z_{2p}^*$ and $v \notin Z(Z_{2p})^*\}$, $C = \{(0, w) : w \in Z_{2p}^*$ and $w \in Z(Z_{2p})^*\}$, $D = \{(a, b) : a \in Z_p^*, b \in Z_{2p}^*$ and $b \in Z(Z_{2p})^*\}$ and $E = \{(c, d) : c \in Z_p^*, d \in Z_{2p}^*$ and $d \notin Z(Z_{2p})^*\}$ respectively. Clearly, all the elements in A, B, C are non-zero zero-divisors. Let $(a, b) \in D$. Then (a, b) is of the form either (a, p) or (a, q), where $q = 2m$, $1 \le m \le p - 1$. Again let $(0, w) \in C$. Similarly, $(0, w)$ is of the form either $(0, p)$ or $(0, q)$, where $q = 2m$, $1 \le m \le p - 1$. Now p/p and $2/q$. This gives $2p/pq$. Therefore, $(a, p)\,(0, q) = (0, 0)$ and $(a, q)\,(0, p) = (0, 0)$. Hence, every element of D is a non-zero zero-divisor. But product of any two elements of E is not equal to zero. Also, product of any element of E with any element of A, B, C and D is not equal to zero because, $cu \neq 0$ for $c, u \in Z_p^*$, $dv \neq 0$ for $d, v \in Z_{2p}^*$ and $d, v \notin Z(Z_{2p})^*$, $dw \neq 0$ for $d, w \in Z_{2p}^*$ and $d \notin Z(Z_{2p})^*$, $w \in Z(Z_{2p})^*$ and $ca \neq 0$ for $c, a \in Z_p^*$ respectively. So, no element of E is a non-zero zero-divisor. Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $Z(R_2)^*$.Then $Z(R_2)^*$ can be partitioned into four disjoint sets A, B, C and D. Now using the **Lemma 3.1** we have $|A| = |Z_p^*| = p - 1$, $|B| = |Z_{2p}^*| - |Z(Z_{2p})^*| = (2p-1) - p = p - 1$, $|C| = |Z(Z_{2p})^*| = p$, $|D| = |Z_p^*|\,|Z(Z_{2p})^*| = (p - 1)\,p = p^2 - p$.

Therefore, $|Z(R_2)^*| = |A| + |B| + |C| + |D| = (p - 1) + (p - 1) + p + (p^2 - p) = p^2 + 2p - 2$

So, the number of vertices of $G = \Gamma(R_2)$ is $p^2 + 2p - 2$.

Let $s = (u, 0)$ be any vertex of A.

(i) Every vertex of A is adjacent to every vertex of B. So, s is adjacent to $p - 1$ vertices of B.

(ii) Every vertex of A is adjacent to every vertex of C. So, s is adjacent to p vertices of C.

(iii) Any vertex of A is not adjacent to any vertex of D as $ua \neq 0$ for $u, a \in Z_p^*$.

Therefore, $deg_G(s) = (p - 1) + p = 2p - 1$.

Let $t = (0, v)$ be any vertex of B.

(i) Every vertex of B is adjacent to every vertex of A. So, t is adjacent to $p - 1$ vertices of A

(ii) Any vertex of B is not adjacent to any vertex of C as $vw \neq 0$ for and $v, w \in Z_{2p}^*$ and $v \notin Z(Z_{2p})^*$, $w \in Z(Z_{2p})^*$

(iii) Any vertex of B is not adjacent to any vertex of D as $vb \neq 0$ for $v, b \in Z_{2p}^*$ and $v \notin Z(Z_{2p})^*$, $b \in Z(Z_{2p})^*$.

Therefore, $deg_G(t) = p - 1$

Let $x = (0, w)$ be any vertex of C. Then either $x = (0, p)$ or $x = (0, q)$, where $q = 2m$, $1 \le m \le p - 1$

(i) Every vertex of C is adjacent to every vertex of A. So, x is adjacent to $p - 1$ vertices of A.

(ii) Case 1: If $x = (0, p)$, then it is adjacent to $p - 1$ vertices of C.

Case 2: If $x = (0, q)$, then it is adjacent to only one vertex of C.

(iii) Case 1: If $x = (0, p)$, then it is adjacent to $|Z_p^*|^2 = (p - 1)^2$ vertices of D.

Case 2: If $x = (0, q)$, then it is adjacent to $|Z_p^*| = p - 1$ vertices of D.

(iv) Any vertex of C is not adjacent to any vertex of B as $wv \neq 0$ for $w, v \in Z_{2p}^*$ and $w \notin Z(Z_{2p})^*$, $v \in Z(Z_{2p})^*$.

Therefore, if $x = (0, p)$, then $deg_G(x) = (p - 1) + (p - 1) + (p - 1)^2 = p^2 - 1$ and if $x = (0, q)$, then $deg_G(x) = (p - 1) + 1 + (p - 1) = 2p - 1$.

Let $y = (a, b)$ be any vertex of D. Then either $y = (a, p)$ or $y = (a, q)$, where $a \in Z_p^*$ and $q = 2m$, $1 \le m \le p - 1$

(i) Case 1: If $y = (a, p)$, then it is adjacent to $p - 1$ vertices of C.

Case 2: If $y = (a, q)$, then it is adjacent to only one vertex of C.

(ii) Any vertex of D is not adjacent to any vertex of A as $au \neq 0$ for $u, a \in Z_p^*$

(iii) Any vertex of D is not adjacent to any vertex of B as $bv \neq 0$ for $b, v \in Z_{2p}^*$ and $b \in Z(Z_{2p})^*$, $v \notin Z(Z_{2p})^*$.

Therefore, if $y = (a, p)$, then $deg_G(y) = p - 1$ and if $y = (a, q)$, then $deg_G(y) = 1$.

Hence, we have $\Delta(G) = p^2 - 1$ and $\delta(G) = 1$.

Theorem 3.3: Let M_2 be of the adjacency matrix for the zero-divisor graph $G = \Gamma(R_2)$ of $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number). Then (i) determinant of M_2 is zero (ii) M_2 is symmetric and singular.

Proof: Follows from **Theorem 2.3.**

Theorem 3.4: Let R_2 be a finite commutative ring such that $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number). Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $V = Z(R_2)^*$.Then $n_G(V) = 2p + \Delta(G) - \delta(G)$, where $n_G(V)$ is the neighborhood number, $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G respectively.

.**Proof:** Let R_2 be a finite commutative ring such that $R_2 = Z_p \times Z_{2p}$ (p is an odd prime number). Let $G = \Gamma(R_2)$ be the zero-divisor graph with vertex set $V = Z(R_2)^*$. Since, $G = \Gamma(R_2)$ is connected [1], we have $n_G(V) = |N_G(V)| = |V| = |Z(R_2)^*|$.

But from **Theorem 3.2**, we have $|Z(R_2)^*| = p^2 + 2p - 2$. Therefore, $n_G(V) = p^2 + 2p - 2$.This implies $n_G(V) = 2p + (p^2 - 1) - 1$. Also $\Delta(G) = p^2 - 1$ and $\delta(G) = 1$ [from **Theorem 3.2**]. This gives $n_G(V) = 2p + \Delta(G) - \delta(G)$.

Remark: If $p = 2$, then $R_2 = Z_2 \times Z_4$. So, this case coincides with the ***case 1*** of **section 2**.

4. CONSTRUCTION OF ZERO -DIVISOR GRAPH FOR $R_3 = Z_p \times Z_{p^2-2}$ (FOR THAT ODD PRIME *p* FOR WHICH $p^2 - 2$ IS A PRIME NUMBER):

Thirdly, we construct the zero-divisor graph for the ring $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2- 2$ is a prime number) and analyze the graph. We start with the cases $p = 3$ and $p = 5$ and then generalize the cases.

Case1: When $p = 3$ we have $R_3 = Z_3 \times Z_7$.
The ring R_3 has 8 non-zero zero-divisors. In this case $V = Z(R_3)^* = \{(1,0), (2,0), (0,1), (0,2),(0,3), (0,4), (0,5),(0,6)\}$ and the zero-divisor graph $G = \Gamma(R_3)$ is given by:

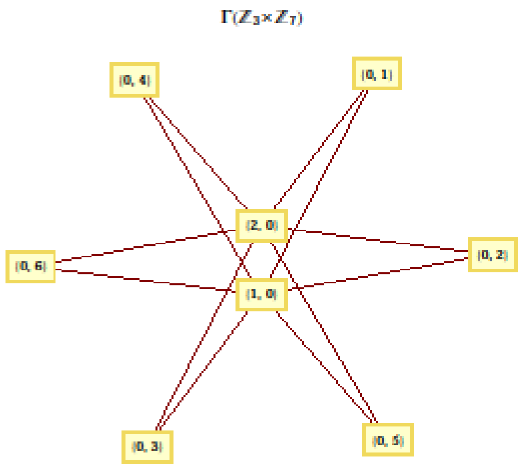

Fig: 5

The closed neighborhoods of the vertices are $N_G[(1,0)] = \{(0,1), (0,2), (0,3), (0,5), (0,6), (1,0)\}$, $N_G[(2,0)] = \{(0,1), (0,2), (0,3), (0,5),(0,6),(2,0)\}$, $N_G[(0,1)] = \{(1,0), (2,0), (0,1)\}$, $N_G[(0,2)] = \{(1,0), (2,0), (0,2)\}$, $N_G[(0,3)] = \{(1,0), (2,0), (0,3)\}$, $N_G[(0,4)] = \{(1,0), (2,0),(0,4)\}$, $N_G[(0,5)]=\{(1,0), (2,0),(0,5)\}$, $N_G[(0,6)] =\{(1,0),(2,0),(0,6)\}$. The neighborhood of V is given by $N_G(V) = \{(1,0), (2,0), (0,1), (0,2), (0,3), (0,4), (0,5),(0,6)\}$.The maximum degree is $\Delta(G) = 6$ and minimum degree is $\delta(G) = 2$. The adjacency matrix for the zero-divisor graph of $R_3 = Z_3 \times Z_7$ is $M_3 = \begin{bmatrix} O_{2\times2} & A_{2\times6} \\ A^T{}_{6\times2} & O_{6\times6} \end{bmatrix}_{8\times8}$ where all the entries of $A_{2\times6}$ is 1, $A^T{}_{6\times2}$ is the transpose of $A_{2\times6}$ and $O_{2\times2}, O_{6\times6}$ are the zero matrices.

Properties of adjacency matrix M_3:

(i) The determinant of the adjacency matrix M_3 corresponding to $G = \Gamma(R_3)$ is 0.
(ii) The rank of the adjacency matrix M_3 corresponding to $G = \Gamma(R_3)$ is 2.
(iii) The adjacency matrix M_3 corresponding to $G = \Gamma(R_3)$ is symmetric and singular.

Case2: When $p = 5$ we have $R_3 = Z_5 \times Z_{23}$.
The ring R_3 has 26 non-zero zero-divisors. In this case $V = Z(R_3)^* = \{(1,0), (2,0), (3,0), (4,0), (0,1),(0,2),(0,3),(0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (0,10), (0,11), (0,12), (0,13), (0,14), (0,15), (0,16), (0,17), (0,18), (0,19),(0,20), (0,21), (0,22)\}$ and the zero-divisor graph $G = \Gamma(R_3)$ is given by:

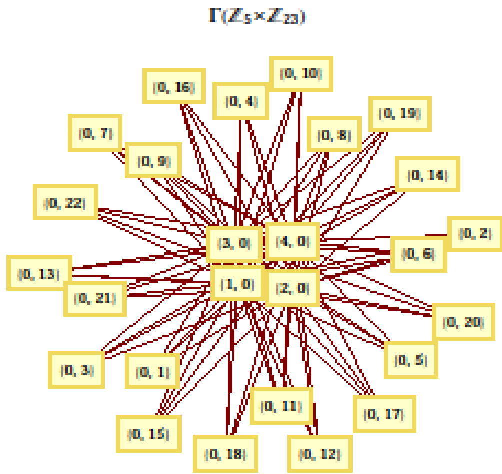

Fig: 6

The closed neighborhoods of the vertices are $N_G[(1,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (0,10), (0,11), (0,12), (0,13), (0,14), (0,15), (0,16), (0,17), (0,18)\}, (0,19), (0,20), (0,21),(0,22),(1,0)\}$, $N_G[(2,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7),(0,8),(0,9), (0,10), (0,11), (0,12), (0,13), (0,14), (0,15), (0,16), (0,17),(0,18)\}, (0,19), (0,20), (0,21), (0,22), (2,0)\}$, $N_G[(3,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7),(0,8),(0,9), (0,10), (0,11), (0,12), (0,13), (0,14), (0,15), (0,16), (0,17), (0,18)\}, (0,19), (0,20), (0,21), (0,22), (3,0)\}$, $N_G[(4,0)] = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (0,10), (0,11), (0,12), (0,13), (0,14), (0,15), (0,16), (0,17), (0,18)\}, (0,19), (0,20), (0,21), (0,22), (4,0)\}$, $N_G[(0,1)] = \{(1,0), (2,0), (3,0), (4,0), (0,1)\}$, $N_G[(0,2)] = \{(1,0), (2,0), (3,0), (4,0), (0,2)\}$, $N_G[(0,3)] = \{(1,0), (2,0), (3,0), (4,0), (0,3)\}$, $N_G[(0,4)] = \{(1,0), (2,0), (3,0),(4,0),(0,4)\}$, $N_G[(0,5)] = \{(1,0),(2,0),(3,0), (4,0),(0,5)\}$, $N_G[(0,6)] = \{(1,0), (2,0),(3,0),(4,0),(0,6)\}$, $N_G[(0,7)] = \{(1,0), (2,0), (3,0), (4,0), (0,7)\}$, $N_G[(0,8)] =\{(1,0),(2,0),(3,0),(4,0), (0,8)\}$, $N_G[(0,9)] = \{(1,0), (2,0), (3,0),(4,0),(0,9)\}$, $N_G[(0,10)] =\{(1,0), (2,0), (3,0), (4,0),(0,10)\}$, $N_G[(0,11)] = \{(1,0), (2,0), (3,0), (4,0), (0,11)\}$, $N_G[(0,12)] = \{(1,0),(2,0), (3,0), (4,0), (0,12)\}$, $N_G[(0,13)] = \{(1,0), (2,0), (3,0),(4,0),(0,13)\}$, $N_G[(0,14)] =\{(1,0), (2,0), (3,0), (4,0),(0,14)\}$, $N_G[(0,15)] = \{(1,0), (2,0), (3,0), (4,0), (0,15)\}$, $N_G[(0,16)] = \{(1,0), (2,0), (3,0), (4,0), (0,16)\}$, $N_G[(0,17)] = \{(1,0), (2,0), (3,0), (4,0), (0,17)\}$, $N_G[(0,18)] = \{(1,0), (2,0), (3,0), (4,0), (0,18)\}$, $N_G[(0,19)] = \{(1,0), (2,0), (3,0),(4,0),(0,19)\}$, $N_G[(0,20)] = \{(1,0), (2,0), (3,0), (4,0), (0,20)\}$, $N_G[(0,21)] = \{(1,0), (2,0), (3,0), (4,0), (0,21)\}$, $N_G[(0,22)] =\{(1,0), (2,0), (3,0), (4,0), (0,22)\}$. The neighborhood of V is given by $N_G(V) = \{(1,0), (2,0), (3,0), (4,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (0,10), (0,11), (0,12), (0,13),(0,14), (0,15), (0,16), (0,17), (0,18), (0,19), (0,20), (0,21),(0,22)\}$. The maximum degree is $\Delta(G) = 22$ and minimum degree is $\delta(G) = 4$. The adjacency matrix for the zero-divisor graph of $R_3 = Z_5 \times Z_{23}$ is

$$M_3 = \begin{bmatrix} O_{4\times 4} & A_{4\times 22} \\ A^T{}_{22\times 4} & O_{22\times 22} \end{bmatrix}_{26\times 26}$$

where all the entries of $A_{4\times 22}$ is 1, $A^T{}_{22\times 4}$ is the transpose of $A_{4\times 22}$ and $O_{4\times 4}$, $O_{22\times 22}$ are the zero matrices.

Properties of adjacency matrix M_3:

(i) The determinant of the adjacency matrix M_3 corresponding to $G = \Gamma(R_3)$ is 0.
(ii) The rank of the adjacency matrix M_3 corresponding to $G = \Gamma(R_3)$ is 2.
(iii) The adjacency matrix M_3 corresponding to $G = \Gamma(R_3)$ is symmetric and singular.

Generalization for $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which p^2–2 is a prime number):

Theorem 4.1: Let R_3 be a finite commutative ring such that $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2 - 2$ is a prime number). Let $G = \Gamma(R_3)$ be the zero-divisor graph with vertex set $Z(R_3)^*$. Then number of vertices of $G = \Gamma(R_3)$ is $p^2 + p - 4$, $\Delta(G) = p^2 - 3$ and $\delta(G) = p - 1$.

Proof: Let R_3 be a finite commutative ring such that that $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2 - 2$ is a prime number). Let $R_3^* = R_3 - \{0\}$. Then R_3^* can be partitioned into disjoint sets A, B and C such that $A = \{(u, 0): u \in Z_p^*\}$, $B = \{(0, v): v \in Z^*{}_{p^2-2}\}$ and $C = \{(a, b) : a \in Z_p^*$ and $b \in Z^*{}_{p^2-2}\}$ respectively. Clearly, all the elements of A and B are non-zero zero-divisors. But product of any two elements of C is not equal to zero. Also, product of any element of C with any element of A and B is not equal to zero because, $au \neq 0$ for $a, u \in Z_p^*$, $bv \neq 0$ for $b, v \in Z^*{}_{p^2-2}$ respectively. So, no element of C is a non-zero zero-divisor. Let $G = \Gamma(R_3)$ be the zero-divisor graph with vertex set $Z(R_3)^*$.Then $Z(R_3)^*$ can be partitioned into two disjoint sets A and B. Now, $|A| = |Z_p^*| = p - 1$ and $|B| = |Z^*{}_{p^2-2}| = p^2 - 3$. Therefore, $|Z(R_3)^*| = |A| + |B| = |Z_p^*| + |Z^*{}_{p^2-2}| = (p - 1) + (p^2 - 3) = p^2 + p - 4$. So, the number of vertices of $G = \Gamma(R_3)$ is $p^2 + p - 4$.

Let $x = (u, 0)$ be any vertex of A.
(i) Every vertex of A is adjacent to every vertex of B. So, x is adjacent to $p^2 - 3$ vertices of B. Therefore, $deg_G(x) = p^2 - 3$.

Let $y = (0, v)$ be any vertex of B.
(i) Every vertex of B is adjacent to every vertex of A. So, y is adjacent to $p - 1$ vertices of A. Therefore, $deg_G(y) = p - 1$.

Hence, we have $\Delta(G) = p^2 - 3$ and $\delta(G) = p - 1$.

Theorem 4 .2: Let M_3 be of the adjacency matrix for the zero-divisor graph $G = \Gamma(R_3)$ of $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2 - 2$ is a prime number). Then (i) determinant of M_3 is zero (ii) M_3 is symmetric and singular.

Proof: Follows from **Theorem 2.3.**

Theorem 4.3: Let R_3 be a finite commutative ring such that $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2 - 2$ is a prime number). Let $G = \Gamma(R_3)$ be the zero-divisor graph with vertex set $V = Z(R_3)^*$. Then $n_G(V) = \Delta(G) + \delta(G)$, where $n_G(V)$ is the neighborhood number, $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G respectively.

Proof: Let R_3 be a finite commutative ring such that $R_3 = Z_p \times Z_{p^2-2}$ (for that odd prime p for which $p^2 - 2$ is a prime number). Let $G = \Gamma(R_3)$ be the zero-divisor graph with vertex set $V = Z(R_3)^*$. Since, $G = \Gamma(R_3)$ is connected [1], we have $n_G(V) = |N_G(V)| = |V| = |Z(R_3)^*|$. But from **Theorem 4.1**, we have $|Z(R_3)^*| = p^2 + p - 4$. Therefore, $n_G(V) = p^2 + p - 4$. This implies $n_G(V) = (p^2 - 3) + (p - 1)$. Also, $\Delta(G) = p^2 - 3$ and $\delta(G) = p - 1$ [from **Theorem 4.1**]. This gives $n_G(V) = \Delta(G) + \delta(G)$.

Remark: If $p = 2$, then $R_3 = Z_2 \times Z_2$. In this case $V = Z(R_3)^* = \{(0, 1),(1, 0)\}$ and $G = \Gamma(R_3)$ is a 1- regular graph. Also, $n_G(V) = 2 = 2\,\Delta(G) = 2\,\delta(G)$.

5. DEFINITIONS AND RELATIONS:

Let R be a commutative ring with unity and let $a \in R$. Then annihilator of a is denoted by $ann(a)$ and defined by $ann(a) = \{x \in R : ax = 0\}$. Let $ann^*(a) = \{x\ (\neq 0) \in R: ax = 0\}$.

The degree of a vertex v of a graph G denoted by $deg(v)$ is the number of lines incident with v.

Given a zero-divisor graph $\Gamma(R)$ with vertex set $Z(R)^*$, then degree of a vertex v of $\Gamma(R)$ is given by $deg(v) = |ann^*(v)|$.

Let A and B be two commutative rings with unity. Then the direct product $A \times B$ of A and B is also a commutative ring with unity.

Let G be a graph and $V(G)$ be the vertex set of G. Let $a, b \in V(G)$. We define a relation $\mathscr{R}$ on $V(G)$ as follows.
For $a, b \in V(G)$, a is related to b under the relation $\mathscr{R}$ if and only if a and b are not adjacent and for any $x \in V(G)$, a and x are adjacent if and only if b and x are adjacent. We denote this relation by $a\mathscr{R}b$.

6. Results of annihilators on $\Gamma(A\times B)$:

Theorem 6.1: The relation $\mathscr{R}$ is an equivalence relation on $V(G)$, where G is any graph.

Proof: For every $a \in V(G)$, we have $a\mathscr{R}a$, as G has no self-loop. For $a, b \in V(G)$, $a\mathscr{R}b$, then clearly, $b\mathscr{R}a$. Again let

$a\mathcal{R}b$ and $b\mathcal{R}c$. If possible suppose, a and c are adjacent. Then we have b and c are also adjacent, a contradiction. So, a and c are not adjacent. Also for $x\in V(G)$, a and x are adjacent $\Leftrightarrow$ b and x are adjacent $\Leftrightarrow$ c and x are adjacent. Therefore, $a\mathcal{R}c$. Hence, the relation $\mathcal{R}$ is an equivalence relation on $V(G)$.

Theorem 6.2: For distinct $a, b\in Z(A\times B)^*$, $a\mathcal{R}b$ in $\Gamma(A\times B)$ if and only if $ann(a)-\{a\} = ann(b)-\{b\}$. Moreover, if $a\mathcal{R}b$ in $\Gamma(R_1\times R_2)$, then $ann(a_1)-\{a_1\} = ann(b_1)-\{b_1\}$ and $ann(a_2)-\{a_2\} = ann(b_2)-\{b_2\}$, where $a = (a_1,a_2)$, $b = (b_1,b_2)$, $a_1,a_2\in A$ and $b_1,b_2\in B$.

Proof: First suppose, for distinct $a, b\in Z(A\times B)^*$, $a\mathcal{R}b$ in $\Gamma(A\times B)$. Let $x\in ann(a)-\{a\}$. This gives $ax = 0$, $a\neq x$. So, a and x are adjacent. Since $a\mathcal{R}b$ we have b and x are adjacent. Therefore, we have $bx = 0$, $b\neq x$. Hence, $x\in ann(b)-\{b\}$. This implies $ann(a)-\{a\}\subseteq ann(b)-\{b\}$. Similarly, $ann(b)-\{b\}\subseteq ann(a)-\{a\}$. This gives $ann(a)-\{a\} = ann(b)-\{b\}$.

Conversely suppose, $ann(a)-\{a\} = ann(b)-\{b\}$. Assume that a and b are adjacent. This gives $ab = 0 \Rightarrow b\in ann(a)-\{a\} = ann(b)-\{b\}$, a contradiction. So, a and b are not adjacent. Again for $x\in Z(A\times B)^*$, a and x are adjacent $\Leftrightarrow$ $ax = 0 \Leftrightarrow x\in ann(b)-\{b\} \Leftrightarrow bx = 0 \Leftrightarrow b$ and x are adjacent. This gives $a\mathcal{R}b$ in $\Gamma(A\times B)$.

If $a = (a_1,a_2)$, $b = (b_1,b_2)\in Z(A\times B)^*$, let $x\in ann(a)-\{a\}$, where $x = (x_1,x_2)$. Then $ax = 0$, $a\neq x$. Therefore, a and x are adjacent. Since $a\mathcal{R}b$ in $\Gamma(A\times B)$, we have a and x are adjacent $\Leftrightarrow$ b and x are adjacent. So, $ax = 0 \Leftrightarrow bx = 0$. This gives $a_1x_1 = 0$, $a_2x_2 = 0 \Leftrightarrow b_1x_1 = 0$, $b_2x_2 = 0$. Hence we have $a_1x_1 = 0 \Leftrightarrow b_1x_1 = 0$, $(x_1\neq a_1,b_1)$ in A and $a_2x_2 = 0 \Leftrightarrow b_2x_2 = 0$ $(x_2\neq a_2,b_2)$ in B. Therefore, $ann(a_1)-\{a_1\} = ann(b_1)-\{b_1\}$ and $ann(a_2)-\{a_2\} = ann(b_2)-\{b_2\}$.

Example 6.3: Consider the commutative ring $Z_2\times Z_4 = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)\}$ and zero-divisor graph $\Gamma(Z_2\times Z_4)$. Here $Z(Z_2\times Z_4)^* = \{(0,1), (0,2), (0,3), (1,0), (1,2)\}$.The possible edges are $\{(0, 1), (1,0)\}$, $\{(0,2), (1,0)\}$, $\{(0,3), (1,0)\}$ and $\{(0,2), (1,2)\}$. The pairs $\{(0, 1), (0, 2)\}$, $\{(0, 1), (0, 3)\}$, $\{(0, 2), (0, 3)\}$ and $\{(1, 0), (1, 2)\}$ establish the existence of relation $\mathcal{R}$ and **Theorem 6.1:**

7. CONCLUSIONS:

In this paper, we study the adjacency matrix and neighborhood associated with zero-divisor graph for direct product of finite commutative rings. Neighborhoods may be used to represent graphs in computer algorithms, via the adjacency list and adjacency matrix representations. Neighborhoods are also used in the clustering coefficient of a graph, which is a measure of the average density of its neighborhoods. In addition, many important classes of graphs may be defined by properties of their neighborhoods.

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